Symmetric inequality with asymmetric condition of equality.

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Let a,b,c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}+\frac{64}{a+b+c}\geq 34.$$

Solution by Arkady Alt, San Jose ,California, USA.

We will prove the inequality for $a,b,c\geq 0$ such that $ab+bc+ca=1$.

Let $s:=a+b+c, q:=abc, p:=ab+bc+ca=1$.

We have $\sum\frac{a^3}{b+c}=\frac{\sum a^3(a+b)(c+a)}{(a+b)(b+c)(c+a)}$. Since $\sum a^3(a+b)(c+a)=$ $\sum(a^4b+ab^4+abc^3+a^5)=\sum ab(a^3+b^3+c^3)+\sum a^5=\sum a^3+\sum a^5$,

and $(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc=s-q$,

$\sum a^3=s^3+3q-3s=s^3-3(s-q)$, $\sum a^5=s^5-5s^3+5s+5qs^2-5q=$ $s^5+5(s-q)-5s^2(s-q)$ then $\frac{\sum a^3}{s-q}=\frac{s^3}{s-q}-3$, $\frac{\sum a^5}{s-q}=\frac{s^5}{s-q}+5-5s^2$

and, therefore, $\sum\frac{a^3}{b+c}+\frac{64}{a+b+c}=\frac{s^3}{s-q}-3+\frac{s^5}{s-q}-5s^2+5+\frac{64}{s}=$ $\frac{s^3+s^5}{s-q}-5s^2+\frac{64}{s}+2$.

Since $q\geq 0$, $q\leq\frac{1}{3\sqrt{3}}$ $\left((abc)^{2/3}\leq\frac{ab+bc+ca}{3}=\frac{1}{3}\right)$, $q\leq\frac{1}{3\sqrt{3}}$, $s\geq\sqrt{3}$ $\left((a+b+c)^2\geq 3(ab+bc+ca)=3\right)$, and $\frac{s^5}{s-q}$ increase by $q\in[0,s)$ then

$$\sum\frac{a^3}{b+c}+\frac{64}{a+b+c}-34=\frac{s^3+s^5}{s-q}-5s^2+\frac{64}{s}-32\geq$$

$$\frac{s^3+s^5}{s-0}-5s^2+\frac{64}{s}-32=\frac{(s^3+4s^2+8s+16)(s-2)^2}{s}\geq 0$$

with equality iff $q=0, s=2$ that is iff $(a,b,c)\in\{(1,1,0),(1,0,1),(0,1,1)\}$.